PEDAGOGY THAT MAKES (NUMBER) SENSE: A CLASSROOM TEACHING EXPERIMENT AROUND MENTAL MATH

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We report on a classroom teaching experiment around number sensible mental math in a semester-long content course for preservice elementary teachers. We designed, implemented, and revised an instructional sequence aimed at students' development of number sense with regard to mental math. The data corpus included: a number sense test administered pre and post, interviews with 13 students pre and post, students' written work, and the instructor's journal. Analysis of the data suggests that students did develop greater number sense as a result of their participation in classroom activities. Particular pedagogical innovations, such as those involving the use of models for reasoning, seem to have supported students' development of number sense. Results can inform mathematics teaching at various levels.

The development of number sense in students is a widely accepted goal of mathematics instruction (NCTM, 2000). Good number sense is especially essential for elementary teachers. Without it, they are ill-equipped to make sense and take advantage of children's often unorthodox but very number sensible solution strategies. Mental math ability is considered a hallmark of number sense (Sowder, 1992). Much work has been done with the aim of identifying the characteristics exhibited and strategies used by individuals who are skilled at mental math (cf. Reys, Rybolt, Bestgen, & Wyatt, 1980, 1982; Hope & Sherrill, 1987; Markivits & Sowder, 1994). Of note is *flexibility* in thinking about numbers and operations (Sowder, 1992).

In two classes of preservice elementary teachers enrolled in a mathematics course focused on Number & Operations, we conducted a classroom teaching experiment (Cobb, 2000) in which the instructor attempted to foster students' development of number sense with regard to mental math. In previous courses, the instructor had found that such development had not occurred. How could one design a class that supported the development of number sense? We undertook a classroom teaching experiment in the paradigm of Design Research (Stephan, 2003) to answer this question. The first author developed a hypothetical learning trajectory (HLT) aimed at students developing the characteristics of skilled mental calculators and estimators. We found that students developed greater number sense with regard to mental math. In addition, the actual learning trajectory that resulted can inform future pedagogy.

We report here on the results of the classroom teaching experiment. This work was part of a larger study, which constituted the first author's Master's thesis**.** In this paper, we focus specifically on the role of models in supporting students' development of number sense.

Theoretical Perspective

Our theoretical orientation can be characterized as sociocultural. Students' individual mathematical activity is recognized as taking place in a social context, while the social environment of the classroom is constituted by collective mathematical activity. As such, the instructor concerned himself with the negotiation of norms and practices (Cobb, 2000).

Reys & Yang (1998) state that "[n]umber sense refers to a person's general understanding of number and operations" and "includes the ability and inclination to use this understanding in

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flexible ways to make mathematics judgments and to develop useful strategies for handling numbers and operations" (p. 226). The perspective on number sense that we take is rooted in Greeno (1991)'s metaphor of situated knowing in a conceptual domain. Thus, the instructor focused on providing students with experiences that would enrich their ability to navigate that domain. According to Greeno (1991), in the environment/model view, "the main capability that we want students to acquire involves constructing and reasoning within models" (p. 212). The use of models became an important aspect of the classroom activity around mental math.

Setting

This study was conducted with undergraduates at a large, urban university in the United States. The participants were preservice elementary teachers enrolled in two sections of a first semester mathematics course, belonging to a four-course sequence. Of the 50 students who agreed to participate in the study, 42 were female. The first author was the teacher of the course. He had taught it for two prior semesters. Basic course topics included quantitative reasoning, place value, meanings for operations, and number sensible mental math. The instructor decided not to treat mental math as an isolated curricular unit but to integrate *authentic* mental math *activity* (Brown, Collins, & Duguid, 1989) throughout the curriculum.

Hypothetical Learning Trajectory

The formulation of the HLT was informed by a review of the literature around number sense and mental math, together with the first author's previous experience teaching the course. We designed an instructional sequence with the goal that students would develop greater number sense with regard to mental math. The HLT was envisioned in terms of three layers. First, the course content was expected to support students' understanding of the mathematics behind particular mental calculative strategies. Second, discrete tasks were devised as a means of assessing students' abilities and the availability to them of various strategies, as well as of providing individual feedback. Students' responses to these tasks would then inform the design of subsequent tasks. Third, mental math activity would be an integral part of problem solving and would provide the occasion for *reflective discourse* around students' strategies (Cobb, Boufi, McClain, & Whitenack, 1997).

The HLT can be briefly articulated in terms of the following sequence of conjectured outcomes:

1) Students recognize opportunities for mental math, both inside and outside the classroom.
2) Students make sense of place value and, as a result, the standard addition and subtraction algorithms.
3) Students make sense of meanings for the operations and consider the use of new mental calculative strategies that build on their understanding.
4) Students confront and make sense of unorthodox strategies and alternative algorithms that are radically different from the ones they know.
5) Students recognize the difference between the use of standard algorithms and tools, such as the empty number line.
6) Students develop their own number sensible mental calculative strategies.

Note that the planned instructional sequence for the HLT was atypical. Typically, an HLT applies to an isolated unit in a curriculum (cf. Gravemeijer, Bowers, & Stephan, 2003; Simon, 1995). In our case, the aspects of classroom instruction that related to number sensible mental math represented a strain of activity that ran through several curricular units.

Data

The data sources drew from classroom events, written artifacts, and individual interviews. Specifically, the data corpus consisted of the following:

a) The instructor's journal, which included accounts of classroom events, as well as rationales for the teaching modifications made during the semester;
b) Students' written work, which included responses to mental math tasks, both in-class and take-home, as well as responses to exam questions;
c) Transcripts of early- and late-semester clinical interviews with 13 students;
d) An adapted version of the Number Sense Rating Scale (Hsu, Yang, & Li, 2001), used as a quantitative measure of number sense, which was administered to students at both the beginning and end of the semester.

Methodology

The general design of this study was that of Design Research, which is characterized by the reflective relationship between classroom-based research and instructional design encompassed in the Design Cycle (Stephan, 2003). As such, data analysis involved three distinct phases:

Phase 1. During the course of the semester, the instructor engaged in formative analysis, in which the instructional sequence was revised in accordance with his interpretations of classroom events and written records of student thinking.

Phase 2. At the end of the semester, data from the number sense test and individual interviews was analyzed in order to assess the effect of the program of instruction on students' number sense. Interviews were structured and task-based (Goldin, 2000). Students were asked to solve one-step story problems mentally and to describe their thinking. Analysis of interviews was interpretive (Clement, 2000), seeking to identify the variety of mental calculative strategies students had employed.

Phase 3. Having noted students' improved number sense, the authors conjectured that certain features of the classroom activity had been particularly significant in supporting that development. These were then analyzed in terms of relevant theoretical constructs.

Results

Significant, selected results are presented here in terms of the three phases of analysis:

Phase 1. The instructor's interpretations of classroom events led to alterations to the instructional approach. A very significant alteration came about in the course of a particular teaching episode. The instructor made immediate innovations to address a local learning goal. Subsequently, aspects of the instruction related to mental math were altered as a result of the instructor's reflections on the episode.

The Teaching Episode spanned four class meetings. On the first day, four interpretations, or distinct meanings, for multiplication were discussed. (Rectangular array/area is one such interpretation.) On the second day, the instructor selected a homework problem for discussion, the solution to which required computation of 26 x 26. As was typical of the integration of mental math activity into the classroom instruction, the instructor asked students to compute this product mentally. A few students shared their solution strategies, which were discussed amongst the class. These suggestions (e.g. 20 x 20 + 6 x 6) seemed to point to a lack of understanding of the origins of partial products in multi-digit multiplication. Although only a few students made such suggestions, no student managed to refute any one of them in sense-making fashion. The instructor suggested making use of a meaning for multiplication and made drawings of rectangles segmented place-value-wise. Students accepted this application of rectangular area and were then able to decide on correct and incorrect solutions. However, the instructor was dissatisfied with this outcome. Students had not made sense of the matter themselves. They had not thought meaningfully about multiplication.

The outcome of the next day was similar. Students' answers to an estimation question again suggested that they were not thinking in terms of partial products. Again, the instructor made sketches of rectangles to help students settle their questions. These were guided more by students' suggestions than had been the case the day before. Still, however, students had not seemed to think meaningfully about multiplication on their own. They would need to understand the origins of partial products in order to reason about mental multiplication strategies. This became the local learning goal.

The instructor designed a Geometer's Sketchpad sketch and a short lesson around it. The sketch was a dynamic representation of a rectangle, segmented place-value-wise, with the areas of the partial rectangles shown. The lesson involved students being asked a sequence of challenging questions related to estimation of products. Students' conjectures were confirmed or refuted either by the sketch, by drawings of rectangles, or by students' arguments. In this context, students began to reason with rectangular area as a *model for* products (Gravemeijer, Bowers, & Stephan, 2003).

Results on a midterm question connected to the sketch lesson were exceptional. Students seemed to have made an important connection between rectangular area and partial products, as well as acquired powerful tools for estimating products. This episode precipitated an important alteration to the greater instructional sequence. The instructor recognized that connections between mainstream course content and applications to mental math were nontrivial. He would need to address the process by which explicit connections could be made between the two. Models came to be emphasized as a means to that end. Rectangular area, in particular, represented an unanticipated tool, which became central to the collective activity around mental multiplication strategies. They empty number line had been used similarly for reasoning about mental addition and subtraction strategies.

Classroom discourse around students' strategies after the Teaching Episode emphasized reasoning with models. For example, commonly seen applications of additive distributivity were characterized in terms of breaking up a rectangle, usually place-value-wise. Figure 1 depicts an example of the strategy that students called "Break up, then make up." In this example, the product of 15 and 24 is represented as a 15-by-24 unit rectangle. Initially, the value of this product is unknown. By breaking up the rectangle conveniently, it is shown to consist of two readily known products, the sum of which gives the total product.

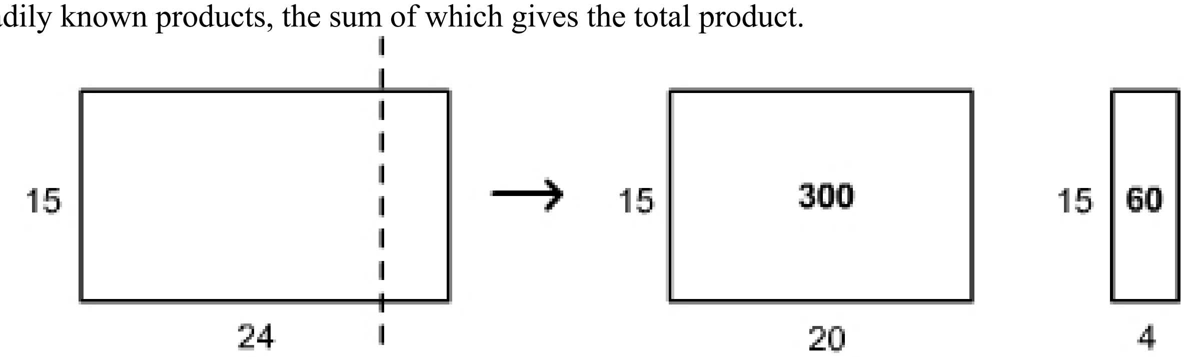

Figure 1. "Break up, then make up."

As the use of models is the focus of this paper, we only mention significant results with regard to the other two layers of the instructional approach. The discrete tasks were deemphasized due to lack of practicality. Mental math activity evolved over the course of the semester. The instructor sought from the beginning to engage students in reflective discourse. Mental math activity seemed lacking until the practice of naming strategies was introduced. Naming facilitated reflective discourse.

Phase 2. Though not the focus of this paper, we note that we found strong evidence in the individual interviews, as well as the number sense test, that students developed significantly greater number sense as a result of their participation in classroom activities. Interview subjects' strategies for mental computation of sums, differences, and products were categorized via constant comparative analysis (Creswell, 1998). Six strategies were seen for mental addition, eight for subtraction, and eight for multiplication. In first interviews, most subjects used only one or two distinct strategies for each of the operations. In second interviews, 12 of 13 subjects used three or more addition strategies, 12 of 13 used three or more subtraction strategies, and 10 of 13 used three or more multiplication strategies.

Markivits & Sowder (1994) categorized their subjects' strategies for mental computation of sums, differences, and products in terms of the degree to which each departed from the mental analogue of the standard algorithm (MASA). In this scheme, *Standard* refers to the MASA for a given operation, *Transition* refers to a method that is still tied to the standard algorithm but differs from it, *Nonstandard with no reformulation* refers to a method that is free from the standard algorithm but does not change the given numbers or operation, and *Nonstandard with reformulation* refers to a method in which the problem is altered to make the computation easier. For our purposes, the above taxonomy was used as an organizing framework. Interview subjects' strategies were categorized as *Standard* (S), *Transition* (T), *Nonstandard with no reformulation* (N), or *Nonstandard with reformulation* (N w/R). This allowed for subjects' strategies to be described in terms of number sensibility. Figure 1 shows the frequency of use of strategies from each category in first versus second interviews.

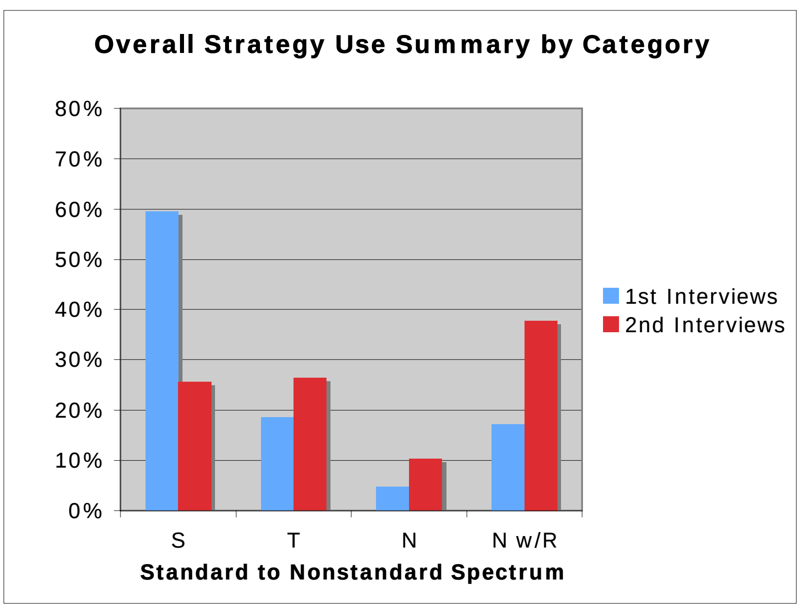

Figure 2. Overall Strategy Use Summary, Pre vs. Post

For each operation, there was a large decrease in the frequency that the MASA was used, accompanied by an increase in use of alternative strategies. Thus, given story problems that called for mental addition, subtraction, and multiplication, subjects exhibited greater flexibility by making use of a wider variety of strategies in second interviews than they had in first interviews. Furthermore, strategies used in second interviews were more number sensible. This is apparent in the movement we see along the spectrum from *Standard* to *Nonstandard*. It is compelling evidence for change in the direction of number sense that *Standard* methods were most common in first interviews, while *Nonstandard with reformulation* became most common in second interviews.

Students also showed significant increases in number sense as measured by the adapted Number Sense Rating Scale. Average scores for the early section increased from 61% to 73%, pretest to posttest. Average scores for the later section increased from 66% to 77%. A total of 48 students took the number sense test both times it was administered They were treated as one group in determining statistical significance. A paired *t*-test was used for a difference of means. Results were statistically significant ($p < 0.005$).

Phase 3. In post-hoc analysis of the instructional sequence, the authors conjectured that the innovations of naming and the use of models had been keys to students' development of number sense. Analysis showed that the classroom discourse around mental math was indicative of reflective discourse and that the practice of naming facilitated *vertical mathematizing* (Freudenthal, 1991). Cobb, Boufi, McClain, & Whitenack (1997) claim that students' participation in reflective discourse "*constitutes conditions for the possibility of mathematical learning*" (p. 264).

The use of models also seemed to be a key to the success of the instructional sequence. Although the use of the empty number line and rectangular area evolved differently, both can be said to have transitioned from a *model of* students' informal activity to a *model for* more formal mathematical reasoning (Gravemeijer, Bowers, & Stephan, 2003). The *model of* to *model for* transition is conjectured to support students' increasingly sophisticated mathematical reasoning. Our use of the empty number line and of rectangular area facilitated students' reasoning more formally about shared mental calculative strategies. In this way, it seems to have supported their development of number sense with regard to mental math.

Continuing the Design Research cycle, the actual learning trajectory that was charted during the classroom teaching experiment informed the construction of a new HLT for the following semester. The new instructional sequence incorporated the practices of naming and the use of models from the start.

Conclusion

The development of number sense in students is an important aim of mathematics instruction. Essential to this goal is that teachers, themselves, have good number sense. In this work, we begin to answer the question of how an instructor can support preservice teachers' development of number sense with regard to mental math. Furthermore, the key practices that emerged in this study can be incorporated into instruction of elementary school students. Analyses such as these can benefit teachers, curriculum developers, and teacher educators. It is also significant that the integration of authentic mental math activity into an existing curriculum supported students' development of number sense without any of the course content being sacrificed.

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